Impact of entanglement on the game-theoretical concept of evolutionary stability

Azhar Iqbal
Department of Mathematics, University of Hull, HU6 7RX, UK

Abstract

An example of a quantum game is presented that explicitly shows the impact of entanglement on the game-theoretical concept of evolutionary stability.

PACS: 03.67.a, 02.50.Le, 87.23.n
Keywords: Quantum Games, Entanglement, Evolutionary stability

1 Introduction

One of the major motivations in recent work on quantum games [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] is to know how quantization of a game affects/changes the existence/location of a Nash equilibrium (NE) [16]. Game theory [17] offers examples where, instead of a unique equilibrium, multiple Nash equilibria emerge as the solutions of a game. In these examples selecting one (or possibly more) from many equilibria requires a refinement of the NE concept. A refinement is a rule/criterion that prefers some equilibria out of many. Numerous refinements are found in game theory; examples include perfect equilibrium (used for extensive- and normal-form games), sequential equilibrium (a fundamental non-cooperative solution concept for extensive-form games) and correlated equilibrium (used for modelling communication among players).

From the view point of quantum games the notion of refinement of NE naturally motivates the question how quantization of a game can affect a refinement; without affecting the corresponding NE. That is, a NE persists[1] in both the classical and quantum versions of a game but its refinement does not. Games and quantization procedures offering such examples can be said to extend the boundary of investigations in quantum games from existence/location of NE to existence/location of one (or more) of its refinements – in relation to quantization of the game. Present paper offers such an example.

The rest of the paper is organized as follows. Section (2) discusses the concept of evolutionary stability and why is can be discussed in quantum games. In Section (3) an example is presented that shows explicitly the role of entanglement in deciding evolutionary stability of a strategy. For comparison a separate subsection discusses the classical case. Section (4) discusses results and indicates lines of further investigation.

2 Evolutionary stability

The concept of an Evolutionarily Stable Strategy (ESS) [18, 19] is the cornerstone of evolutionary game theory [20]; introduced some thirty years ago by mathematical biologists as a game-theoretical model to understand equilibrium behavior of a system of interacting entities. Their original intention was to explain phenomena like polymorphism[2] of behavior in animal societies. These phenomena happen in spite of the fact that the individuals in these societies have neither

[1] For two-player games by saying that a NE persists in both the classical and quantum version of a game means that there exists a NE consisting of quantum strategies that rewards both the players exactly the same as does the corresponding NE in the classical version of the game.

[2] Polymorphism is the occurrence of different forms, stages or types in individual organisms or in organisms of the same species.

conscience; nor rationality; nor expectations; nor even the choice between several behavioral patterns which are thought to be determined genetically. Sometimes, even using the term "individual" for members of these animals has itself been questioned.

Roughly speaking, ESS is described as a strategy having the property that if all members of a population adapt it, no mutant strategy could invade the population under the influence of natural selection. That is, if a strategy is adapted by at least $1 - \epsilon_0$ fraction of the population then it can resist invasion by any $\epsilon < \epsilon_0$ mutant strategies. ESS is generally accepted as another refinement of the NE concept; but the concept has been given mathematical formulations in several contexts [23]. In the familiar context, ESS is described as a refinement on the set of symmetric Nash equilibria which is robust against small changes (mutations) that may appear in prevalent strategy. The robustness against mutations is referred to as *evolutionary stability.*

The definition of an ESS, as it was originally introduced [18] in mathematical biology, assumes [24]:

- An infinite population of players engaged in random pair-wise contests.
- Each player being programmed to play only one strategy.
- An evolutionary pressure ensuring that the better players have better chances of survival.

It is seen that the setting of evolutionary game theory diverges away from the usual setting of game theory in disassociating its players from the capacity to make rational decisions. The players' strategies are inheritable traits (phenotypes) that evolution tests for their suitability and value in the players' struggle for survival over the course of time. In contrast, the usual approach in game theory assumes players both capable of making rational decisions and always interested to maximize their payoffs.

Interestingly, in parallel to its original formulation, the ESS theory can also be interpreted such that a player's strategy is not a phenotype but an available option or a possible state attributable to a player. With this interpretation the *same* players continue their repeated pair-wise contests. The evolutionary pressure, however, now ensures the survival of better-performing strategies, *not* players. ESS will emerge as a strategy which if played by most of the population will withstand invasion from mutant strategies that are played by small number of the members of the population. This view of the ESS concept can be studied also in quantum mechanical versions of evolutionary games because it allows replacement of classical strategies with their quantum analogues. Moreover, the view permits not to associate with the players a capacity to make rational decisions; thus retaining intact one of the important features of the ESS theory.

A question can be raised here: Where in nature do the quantum strategies exist and are being subjected to evolutionary pressures? It seems that example of inter-molecular interactions can be promising candidate. These interactions can both be pair-wise and taking place under evolutionary forces. For these interactions the players' disassociation from their capacity to act rationally can also be granted, without further assumptions.

We now move to a question of interest that such setting is bound to raise: How game-theoretical solution concepts, which are especially developed for the understanding of evolutionary dynamics, adapt/shape/change themselves with players having access to quantum strategies? For the possible situation of inter-molecular interactions this question shapes itself to ask how game-theoretical solution concepts, that are especially developed for evolutionary games, predict different equilibrium states of a population consisting of interacting molecules to which quantum strategies can be associated.

The emerging field of quantum games recognizes entanglement as a resource giving new, and often counter-intuitive dimensions to world of playing games. From the view point of ESS theory the players' sharing of a new resource of entanglement leads one to ask whether entanglement can change the evolutionary stability of a symmetric NE. Of course, during this change the corresponding NE remains intact in both the classical and quantum forms of the game. Evolutionary stability is a solution concept having a relevance to a population as a whole with regards to its capacity to withstand mutant strategies. If entanglement is found to decide evolutionary stability

then the lesson from the ESS theory is that entanglement's effects are not confined to the pair-wise encounters but the phenomenon can also decide the fate of the whole population, in terms of its susceptibility to invasion from the mutant strategies that appear in small numbers.

3 An example

Though recent work in quantum games presents examples [15] of games where evolutionary stability is related to quantization of a game, a direct and explicit relationship between a measure of entanglement and the mathematical concept of evolutionary stability is still to be investigated even for two-player games. Earlier work [15] on this topic uses a particular quantization scheme for matrix games suggested in the Ref. [9]. In this scheme a measure of entanglement does not appear explicitly in players' payoff expressions. In Eisert et al.'s scheme, on the other hand, players' payoffs contain entanglement explicitly, which makes possible, in the present contribution, to develope an example that shows the relationship between entanglement and evolutionary stability.

Consider a symmetric bi-matrix game given by the matrix:

$$\begin{array}{cc} & \text{Bob} \\ \text{Alice} & \begin{array}{cc} & \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{array}{c} S_1 \\ S_2 \end{array} & \left(\begin{array}{cc} (r,r) & (s,t) \\ (t,s) & (u,u) \end{array} \right) \end{array} \end{array} \tag{1}$$

Suppose Alice and Bob play the strategy S_1 with probabilities p and q, respectively. The strategy S_2 is then played with probabilities $(1-p)$ and $(1-q)$ by Alice and Bob, respectively. We denote Alice's payoff by $P_A(p,q)$ when she plays p and Bob plays q. That is, Alice's and Bob's strategies are identified from the numbers $p, q \in [0,1]$, without referring to S_1 and S_2. For the matrix (1) the Alice's payoff $P_A(p,q)$, for example, reads

$$P_A(p,q) = rpq + sp(1-q) + t(1-p)q + u(1-p)(1-q) \tag{2}$$

Similarly, Bob's payoff $P_B(p,q)$ can be written. In this symmetric game we have $P_A(p,q) = P_B(q,p)$ and, without using subscripts, $P(p,q)$, for example, describes the payoff to p-player against q-player. In this game the inequality

$$P(p^*,p^*) - P(p,p^*) \geqslant 0 \tag{3}$$

says that the strategy p^*, played by both the players, is a NE.

3.1 Evolutionary stability: classical game

For symmetric contests an Evolutionarily Stable Strategy (ESS) is defined as follows. A strategy p^* is an ESS when

$$P(p^*,p^*) > P(p,p^*) \tag{4}$$

that is, p^* is a strict NE. If it is not the case and

$$P(p^*,p^*) = P(p,p^*) \text{ then } p^* \text{ is ESS if } P(p^*,p) > P(p,p) \tag{5}$$

showing that every ESS is a NE but not otherwise.

We consider a case when

$$s = t, \quad r = u \text{ and } \quad (r-t) > 0 \tag{6}$$

in the matrix (1). In this case the inequality (3) along with the definition (2) gives

$$P(p^*,p^*) - P(p,p^*) = (p^* - p)(r-t)(2p^* - 1) \tag{7}$$

and the strategy $p^* = 1/2$ comes out as a mixed NE. From the defining inequalities (4,5) we get $P(1/2, 1/2) - P(p, 1/2) = 0$ and the first condition (4) of an ESS does not apply. The second condition (5), then, gives

$$P(1/2, p) - P(p, p) = (r - t)\{2p(1 - p) - 1/2\} \tag{8}$$

which can not be strictly greater than zero given $(r - t) > 0$. For example, at $p = 0$ it becomes a negative quantity. Therefore, for the matrix game defined by (1) and (6) the strategy $p^* = 1/2$ is a symmetric NE, but it is not evolutionarily stable. Also, at this equilibrium both players get $(r + t)/2$ as their payoffs.

3.2 Evolutionary stability: quantum game

Consider the *same* game, i.e. defined by (1) and (6), played now by the set-up proposed by Eisert et al. [4, 5]. This scheme suggests a quantum version of the game (1) by assigning two basis vectors $|S_1\rangle$ and $|S_2\rangle$ in the Hilbert space of a qubit. States of the two qubits belong to two-dimensional Hilbert spaces H_A and H_B respectively. State of the game is defined by a vector residing in the tensor-product space $H_A \otimes H_B$, spanned by the basis $|S_1S_1\rangle, |S_1S_2\rangle, |S_2S_1\rangle$ and $|S_2S_2\rangle$. Game's initial state is $|\psi_{ini}\rangle = \hat{J}\,|S_1S_1\rangle$ where $\hat{J}$ is a unitary operator known to both the players. Alice's and Bob's strategies are unitary operations $\hat{U}_A$ and $\hat{U}_B$, respectively, chosen from a strategic space Ş. The state of the game changes to $\hat{U}_A \otimes \hat{U}_B \hat{J}\,|S_1S_1\rangle$ after players' actions. Finally, measurement consists of applying reverse unitary operator $\hat{J}^\dagger$ followed by a pair of Stern-Gerlach type detectors. Before detection the final state of the game is $|\psi_{fin}\rangle = \hat{J}^\dagger \hat{U}_A \otimes \hat{U}_B \hat{J}\,|S_1S_1\rangle$. The players' expected payoffs can then be written as the projections of the state $|\psi_{fin}\rangle$ onto the basis vectors of tensor-product space $H_A \otimes H_B$, weighed by the constants appearing in the game (1). For example, Alice's payoff reads

$$P_A = r\left|\langle S_1S_1 \mid \psi_{fin}\rangle\right|^2 + s\left|\langle S_1S_2 \mid \psi_{fin}\rangle\right|^2 + t\left|\langle S_2S_1 \mid \psi_{fin}\rangle\right|^2 + u\left|\langle S_2S_2 \mid \psi_{fin}\rangle\right|^2 \tag{9}$$

Bob's payoff is, then, obtained by the transformation $s \rightleftarrows t$ in Eq. (9). Eisert et al. [4, 5] allowed players' actions from the space Ş of unitary operators of the form

$$U(\theta, \phi) = \begin{pmatrix} e^{i\phi}\cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi}\cos(\theta/2) \end{pmatrix} \tag{10}$$

where

$$\theta \in [0, \pi] \text{ and } \phi \in [0, \pi/2] \tag{11}$$

They defined their unitary operator $\hat{J} = \exp\{i\gamma S_2 \otimes S_2/2\}$ with $\gamma \in [0, \pi/2]$ representing a measure of the game's entanglement. At $\gamma = 0$ the game reduces to its classical form.

After this note on Eiset et al.'s scheme, we set $s_A \equiv (\theta_A, \phi_A)$ and $s_B \equiv (\theta_B, \phi_B)$ to denote Alice's and Bob's strategies, respectively. Because the quantum game is symmetric i.e. $P_A(s_A, s_B) = P_B(s_B, s_A)$ we can write, as before, $P(s_A, s_B)$ for the payoff to s_A-player against s_B-player. For quantum form of the game (1,6) one finds

$$P(s_A, s_B) = (1/2)(r - t)\{1 + \cos\theta_A \cos\theta_B + \sin\theta_A \sin\theta_B \sin\gamma \sin(\phi_A + \phi_B)\} + t \tag{12}$$

The definition of a NE gives $P(s^*, s^*) - P(s, s^*) \geqslant 0$ where $s = (\theta, \phi)$ and $s^* = (\theta^*, \phi^*)$. The definition can be written as

$$\{\partial_\theta P \mid_{\theta^*, \phi^*} (\theta^* - \theta) + \partial_\phi P \mid_{\theta^*, \phi^*} (\phi^* - \phi)\} \geq 0 \tag{13}$$

We search for a quantum strategy $s^* = (\theta^*, \phi^*)$ for which both $\partial_\theta P \mid_{\theta^*, \phi^*}, \partial_\phi P \mid_{\theta^*, \phi^*}$ vanish at $\gamma = 0$ and at some other value of γ which is not zero. For the payoffs (12) the strategy $s^* = (\pi/2, \pi/4)$ satisfies these conditions. For this strategy the Eq. (12) gives

$$P(s^*, s^*) - P(s, s^*) = (1/2)(r - t)\sin\gamma\,\{1 - \sin(\phi + \pi/4)\sin\theta\} \tag{14}$$

At $\gamma = 0$ the strategy $s^* = (\pi/2, \pi/4)$, when played by both the players, is a NE which rewards the players same as does the strategy $p^* = 1/2$ in the classical version of the game i.e. $(r + t)/2$. Also, then we have $P(s^*, s^*) - P(s, s^*) = 0$ from Eq. (14) and the ESS's second condition (5) applies. Use Eq. (12) to evaluate

$$P(s^*, s) - P(s, s) = -(r - t)\cos^2(\theta) + (1/2)(r - t)\sin\gamma\sin\theta\,\{\sin(\phi + \pi/4) - \sin\theta\sin(2\phi)\} \tag{15}$$

which at $\gamma = 0$ reduces to $P(s^*, s) - P(s, s) = -(r - t)\cos^2(\theta)$, which can assume negative values. The game's definition (6) and the ESS's second condition (5) show that the strategy $s^* = (\pi/2, \pi/4)$ is not evolutionarily stable at $\gamma = 0$.

Now consider the case when $\gamma \neq 0$ to know about the evolutionary stability of the *same* quantum strategy. From (11) we have both $\sin\theta, \sin(\phi + \pi/4) \in [0, 1]$ and the Eq. (14) indicates that $s^* = (\pi/2, \pi/4)$ remains a NE for all $\gamma \in [0, \pi/2]$. The product $\sin(\phi + \pi/4)\sin\theta$ attains a value of 1 only at $s^* = (\pi/2, \pi/4)$ and remains less than 1 otherwise. The Eq. (14) shows that for $\gamma \neq 0$ the strategy $s^* = (\pi/2, \pi/4)$ becomes a strict NE for which the ESS's first condition (4) applies. Therefore, for the game defined in (6) the strategy $s^* = (\pi/2, \pi/4)$ is evolutionarily stable for a non-zero measure of entanglement γ.

4 Discussion

The above example shows explicitly how presence of entanglement leads to evolutionary stability of a strategy. It is of interest for three apparent reasons. Firstly, the game-theoretical concept of evolutionary stability has very rich literature in game theory, mathematical biology and evolutionary economics [22, 21]. Secondly, the result that the game-theoretical concept of evolutionary stability can be discussed in relation to entanglement opens a new role for this phenomenon. It is a role where entanglement decides whether a population of interacting entities can withstand invasion from mutant strategies appearing in small numbers. Thirdly, this extended role for entanglement can possibly be helpful to better understand entanglement itself.

A possible criticism of studying evolutionary stability in quantum games may come from the following view point. Being a game-theoretical solution concept, originally developed to understand problems in population biology, how can the concept be taken out of its context of population biology and discussed in quantum games? Evolutionary stability was indeed originally introduced within population biology but the concept can also be given an interpretation in term of two-player game that is infinitely repeated. Secondly, the population setting that evolutionary stability assumes does not come only from discussion of the problems of population biology. Even the concept of NE, as it was originally developed by John Nash, assumed a population of players. In his unpublished thesis he wrote '*it is unnecessary to assume that the participants have...... the ability to go through any complex reasoning process. But the participants are supposed to accumulate empirical information on the various pure strategies at their disposal.......We assume that there is a population ....of participants......and that there is a stable average frequency with which a pure strategy is employed by the "average member" of the appropriate population*'[20, 16]. Evolutionary stability, as a game-theoretical concept, also has roots in efforts to get the game theory rid of its usual approach devoting itself to analyzing games among hyper-rational players always ready and engaged in selfish interests to increase their payoffs. The lesson evolutionary stability teaches is that playing games can be disassociated from players' capacity to make rational decisions. Such disassociation seems equally fruitful to possible situations where quantum games are being played in nature; because associating rationality to quantum interacting entities is of even much more remote possibility then bacteria and viruses whose behavior evolutionary game theory explains.

An interesting approach [25] characterizes ESSs in terms of extremal states of a function known as *evolutionary entropy* defined by

$$E = -\sum_i \mu_i \log \mu_i$$

where μ_i represents the relative contribution of the ith strategy to the total payoff. A possible extension of the present approach may be the case when entanglement decides extremal states of evolutionary entropy. Extension on similar lines can be proposed for another quantity which Bomze called *relative negentropy* [24] and it is optimized during the course of evolution.

In the Section (3.2) entanglement gives evolutionary stability to a symmetric NE by making it a strict NE as well. Thus evolutionary stability is achieved by only using the ESS's first condition. Perhaps a more interesting example is when entanglement gives evolutionary stability via the ESS's second condition. That is, entanglement makes $P(s^*, s)$ strictly greater than $P(s, s)$ when $P(s^*, s^*)$ and $P(s, s^*)$ are equal.

References

[1] N. D. Mermin, Am. J. Phys. **58**, 731 (1990).

[2] L. Vaidman, Found. Phys. **29**, 615 (1999).

[3] D. A. Meyer, Phys. Rev. Lett. **82** 1052-1055 (1999).

[4] J. Eisert, M. Wilkens, M. Lewenstein, Phys. Rev. Lett. **83,** 3077 (1999).

[5] J. Eisert and M. Wilkens, J. Mod. Opt. **47**, 2543 (2000).

[6] S. J. van Enk, R. Pike, Phys Rev A **66**, 024306 (2002).

[7] Simon C. Benjamin, Patrick M. Hayden, Phys. Rev. A **64** (3), 030301 (2001).

[8] Simon C. Benjamin, Patrick M. Hayden, Phys. Rev. Lett. **87** (6), 069801 (2001).

[9] L. Marinatto and T. Weber, Phys. Lett. A **272**, 291 (2000).

[10] Jiangfeng Du, Hui Li, Xiaodong Xu, Mingjun Shi, Jihui Wu, Xianyi Zhou, Rongdian Han, Phys. Rev. Lett. **88**, 137902 (2002).

[11] Chiu Fan Lee, Neil Johnson, Phys. Rev. A **67**, 022311 (2003).

[12] E. W. Piotrowski, J. Sladkowski, Int. J. Theor. Phys. **42** 1089 (2003).

[13] Adrian Flitney, PhD thesis, Adelaide University, 2005, URL: `http://thesis.library.adelaide.edu.au/public/adt-SUA20050429.143949/`.

[14] Taksu Cheon, Izumi Tsutsui, `quant-ph/0503233 v2`.

[15] Azhar Iqbal, PhD thesis, Quaid-i-Azam University, 2004, `quant-ph/0503176`.

[16] R. J. Leonard: Reading Cournot, reading Nash: The creation and stabilization of the Nash equilibrium. The Economic Journal **104**, 492-511 (1994).

[17] v. Neumann J. and O. Morgenstern. *Theory of Games and Economic Behaviour*. (Princeton 1953).

[18] J. Maynard Smith, G. R. Price, Nature **246**, 15 (1973).

[19] J. Maynard Smith, *Evolution and the Theory of Games*. (Cambridge Uni. Press, Cambridge, 1982).

[20] Josef Hofbauer and Karl Sigmund. *Evolutionary Games and Population Dynamics.* Cambridge University Press (1998).

[21] Journal of Evolutionary Economics, Springer-Verlag.

[22] Daniel Friedman, On economic applications of game theory. J. Evol. Econ. **8**, 15 (1998) and references within.

[23] S. Lessard, Evolutionary stability: one concept, several meanings, Theoret. Popul. Biol. **37** (1990) 159.

[24] Immanuel M. Bomze, Evolutionary stability is not a foolish game. Cent. Europ. J. Oper. Res. Eco. Vol. **4**, Number 1, 25 (1996).

[25] Lloyd Demetrius, Volker Matthias Gundlach. Game theory and evolution: finite size and absolute fitness measures. Mathematical Biosciences **168** (2000) 9-38.